Sep 21 2015

Exercises with solutions

Ondrej Škopek <oskopek@matfyz.cz>

1. Number magic

Problem

Surely you've seen a similar exercise before:

- Think of a number between 1-5
- Multiply it by 2
- Add 2
- Multiply by 3
- Subtract the double of the original number
- Add 6
- Divide by 4
- Subtract the original number (again)

Answer and show your thinking:

- What's the result?
- Will you always get the same result? Or just for the numbers 1-5?
- Why doesn't it matter what number you start with?

Solution

- What's the result?

  Try it with at least one number.

- Will you always get the same result? Or just for the numbers 1-5?

  We could try all the numbers and compare the results. What if we want to take all the numbers from 1-1000? Or all natural numbers? Or all real numbers? It looks like the result will always be 3. But, are we sure?

- Why doesn't it matter what number you start with?

  Let's try to write down the operations we are doing. Let x be the starting number.

  - Multiply it by 2 $\rightarrow 2x$
  - Add 2 $\rightarrow 2x + 2$
  - Multiply by 3 $\rightarrow (2x + 2) \cdot 3$
  - Subtract the double of the original number $\rightarrow ((2x + 2) \cdot 3) - 2x$
  - Add 6 $\rightarrow (((2x + 2) \cdot 3) - 2x) + 6$
  - Divide by 4 $\rightarrow ((((2x + 2) \cdot 3) - 2x) + 6)/4$
  - Subtract the original number (again) $\rightarrow (((((2x + 2) \cdot 3) - 2x) + 6)/4) - x$

Let's simplify the expression.

$$
\begin{aligned}
&(((((2x + 2) \cdot 3) - 2x) + 6)/4) - x \\
&= ((((((2x + 2) \cdot 3) - 2x) + 6)/4) - x \\
&= (((((6x + 6) - 2x) + 6)/4) - x \\
&= (((4x + 6) + 6)/4) - x \\
&= ((4x + 12)/4) - x \\
&= (x + 3) - x \\
&= 3
\end{aligned}
$$

We took any real number and with the help of legal simplifications simplified it. We saw, that the complex formula we started with is actually only a complicated way of writing the number 3. That means, that whatever we choose x to be, the expression will always yield 3 as it's value and we'll always get 3 as a result of applying these operations.

2. Seven bridges of Königsberg [12]

Problem

The problem was to find a walk through the city that would cross each bridge once and only once. The islands could not be reached by any route other than the bridges, and every bridge must have been crossed completely every time; one could not walk halfway onto the bridge and then turn around and later cross the other half from the other side (the walk need **not** start and end at the same spot). Can you find such a walk? Is there even one?

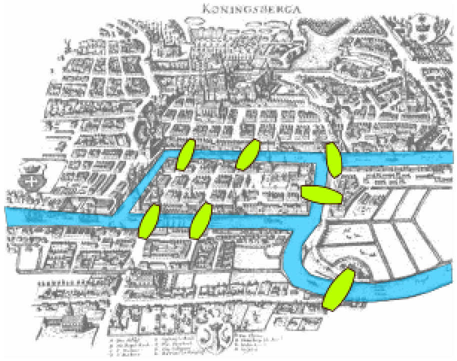
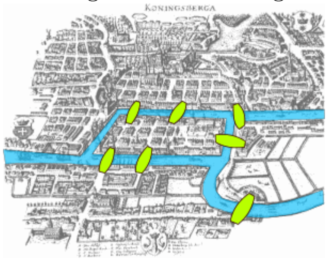
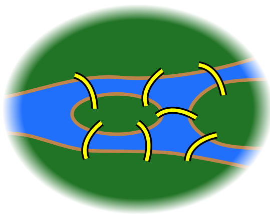
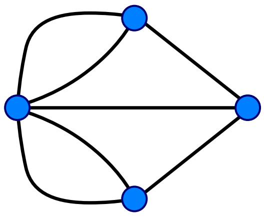


Figure 1: Source: [9]

Solution

Euler proved that the problem has no solution. First, Euler pointed out that the choice of route inside each land mass is irrelevant. The only important feature of a route is the sequence of bridges crossed. This allowed him to reformulate the problem in abstract terms (laying the foundations of graph theory), eliminating all features except the list of land masses and the bridges connecting them. In modern terms, one replaces each land mass with an abstract „vertex” or node, and each bridge with an abstract connection, an „edge”, which serves as a record which pair of vertices is connected by which bridge. The resulting mathematical structure is called a graph.

 [9] →  [5] →  [10]

Since only the connection information is relevant, the shape of pictorial representations of a graph may be distorted in any way, without changing the graph itself. Only the existence (or absence) of an edge between each pair of nodes is significant. For example, it does not matter whether the edges drawn are straight or curved, or whether one node is to the left or right of another.

Next, Euler observed that (except at the endpoints of the walk), whenever one enters a vertex by a bridge, one leaves the vertex by a bridge. In other words, during any walk in the graph, the number of times one enters a non-terminal vertex equals the number of times one leaves it. Now, if every bridge has been traversed exactly once, it follows that, for each land mass (except for the ones chosen for the start and finish), the number of bridges touching that land mass must be even (half of them, in the particular traversal, will be traversed „toward” the landmass; the other half, „away” from it). However, all four of the land masses in the original problem are touched by an odd number of bridges (one is touched by 5 bridges, and each of the other three are touched by 3). Since, at most, two land masses can serve as the endpoints of a putative walk, the proposition of a walk traversing each bridge once leads to a contradiction.

In modern language, Euler shows that the possibility of a walk through a graph, traversing each edge exactly once, depends on the degrees of the nodes. The degree of a node is the number of edges touching it. Euler's argument shows that a necessary condition for the walk of the desired form is that the graph be connected and have exactly zero or two nodes of odd degree. This condition turns out also to be sufficient – a result stated by Euler and later proven by Carl Hierholzer. Such a walk is now called an Eulerian path or Euler walk in his honor. Further, if there are nodes of odd degree, then any Eulerian path will start at one of them and end at the other. Since the graph corresponding to historical Königsberg has four nodes of odd degree, it cannot have an Eulerian path.

An alternative form of the problem asks for a path that traverses all bridges and also has the same starting and ending point. Such a walk is called an Eulerian circuit or an Euler tour. Such a circuit exists if, and only if, the graph is connected, and there are no nodes of odd degree at all. All Eulerian circuits are also Eulerian paths, but not all Eulerian paths are Eulerian circuits.

3. 100 prisoners problem (The Locker Problem) [4]

Problem

The director of a prison offers 100 prisoners on death row, which are numbered from 1 to 100, a last chance. In a room there is a cupboard with 100 drawers. The director puts in each drawer the number of exactly one prisoner in random order and closes the drawers afterwards. The prisoners enter the room one after another. Each prisoner may open and look into 50 drawers in any order and the drawers are closed again afterwards. If during this search every prisoner finds his number in one of the drawers, all prisoners are pardoned. If just one prisoner does not find his number, all prisoners have to die. Before the first prisoner enters the room, the prisoners may discuss their strategy, afterwards no communication of any means is possible. What is the best strategy for the prisoners?

Solution

If every prisoner selects 50 drawers at random, the probability that a single prisoner finds his number is 50%. Therefore, the probability that all prisoners find their numbers is the product of the single probabilities which is $(\frac{1}{2})^{100} \approx 0.0000000000000000000000000000008$, a vanishingly small number. The situation appears hopeless for the prisoners.

Strategy

There exists a better strategy though. The key to success is that the prisoners do not have to decide beforehand which drawers they are going to open. Each prisoner can use the information gained from the contents of previously opened drawers to help him decide which drawer to open next. Another important observation is that this way the success of one prisoner is not independent of the success of the other prisoners. [2]

In order to describe the strategy, not only the prisoners, but also the drawers are numbered from 1 to 100. The strategy is now as follows: [3]

1. Each prisoner first opens the drawer with his own number.
2. If this drawer contains his number he is finished with his search and was successful.
3. Otherwise, the drawer contains the number of another prisoner and he next opens the drawer with this number.
4. The prisoner repeats steps 2 and 3 until he finds his own number or has opened 50 drawers.

This approach ensures that every time a prisoner opens a drawer he either finds his own number or the number of another prisoner he has not encountered so far.

Examples

That this is a promising strategy is illustrated with the following example using eight prisoners and drawers, whereby each prisoner may open four drawers. The prison director has distributed the prisoners' numbers into the drawers in the following fashion

number of drawer	1	2	3	4	5	6	7	8
number of prisoner	7	4	6	8	1	3	5	2

The prisoners now act as follows:

- Prisoner 1 first opens drawer 1 and finds number 7. Then he opens drawer 7 and finds number 5. Then he opens drawer 5 where he finds his own number and is successful.
- Prisoner 2 opens drawers 2, 4, and 8 in this order. In the last drawer he finds his own number 2.
- Prisoner 3 opens drawers 3 and 6, where he finds his own number.
- Prisoner 4 opens drawers 4, 8, and 2 where he finds his own number. An outside observer could have derived this from the information gained by prisoner 2.

- That prisoners 5 to 8 will also find their numbers can also be derived from the information gained by the first three prisoners.

In this case, all prisoners will be successful in finding their numbers. This is, however, not always the case. In the following example the prison director has distributed the numbers like this:

number of drawer	1	2	3	4	5	6	7	8
number of prisoner	3	1	7	5	8	6	4	2

In this case, prisoner 1 opens drawers 1, 3, 7, and 4, at which point he has to stop unsuccessfully. Except for prisoner 6, who directly finds his number, all other prisoners are also unsuccessful.

Permutation representation

The prison director's assignment of prisoner numbers to drawers can mathematically be described as a permutation of the numbers 1 to 100. Such a permutation is a one-to-one mapping of the set of natural numbers from 1 to 100 to itself. A sequence of numbers which after repeated application of the permutation returns to the first number is called a cycle of the permutation. Every permutation can be decomposed into disjoint cycles, that is cycles which have no common elements. The permutation of the first example above can be written in cycle notation as

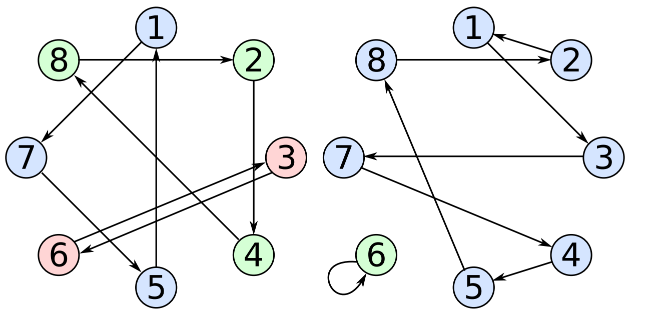


Figure 2: Graph representations of the permutations (1 7 5)(2 4 8)(3 6) and (1 3 7 4 5 8 2)(6). Source: [7], [6]

$$(1\ 7\ 5)(2\ 4\ 8)(3\ 6)$$

and thus consists of two cycles of length 3 and one cycle of length 2. The permutation of the second example is accordingly

$$(1\ 3\ 7\ 4\ 5\ 8\ 2)(6)$$

and consists of a cycle of length 7 and a cycle of length 1. The cycle notation is not unique since a cycle of length l can be written in l different ways depending on the starting number of the cycle. During the opening the drawers in the above strategy, each prisoner follows a single cycle which always ends with his own number. In the case of eight prisoners, this cycle-following strategy is successful if and only if the length of the longest cycle of the permutation is at most 4. If a permutation contains a cycle of length 5 or more, all prisoners whose numbers lie in such a cycle will not have reached their own number after 4 steps.

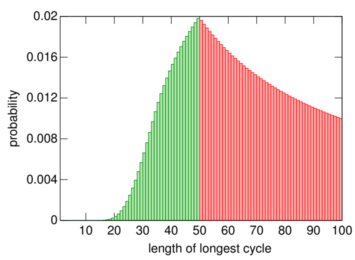


Figure 3: Probability distribution of the length of the longest cycle of a random permutation of the numbers 1 to 100. The green area corresponds to the survival probability of the prisoners. Source: [11]

Probability of success

In the initial problem, the 100 prisoners will be successful if the longest cycle of the permutation has a length of at most 50. Their survival probability is therefore equal to the probability that a random permutation of the numbers 1 to 100 contains no cycle of length greater than 50. This probability is now determined.

A permutation of the numbers 1 to 100 can contain at most one cycle of length $l > 50$. There are exactly $\binom{100}{l}$ ways to select the numbers of such a cycle (see combinations). Within this cycle, these numbers can be arranged in $(l-1)!$ ways since there are l possibilities to select the starting number of the cycle. The remaining numbers can be arranged in $(100-l)!$ ways. Therefore, the number of permutations of the numbers 1 to 100 with a cycle of length $l > 50$ is equal to

$$\binom{100}{l} \cdot (l-1)! \cdot (100-l)! = \frac{100!}{l}$$

The probability, that a (uniformly distributed) random permutation contains no cycle of length greater than 50 is with the formula for single events and the formula for complementary events thus given by

$$1 - \frac{1}{100!}\left(\frac{100!}{51} + \ldots + \frac{100!}{100}\right) = 1 - \left(\frac{1}{51} + \ldots + \frac{1}{100}\right) = 1 - (H_{100} - H_{50}) \approx 0.31183$$

where H_n is the n-th harmonic number. Therefore, using the cycle-following strategy the prisoners survive in a surprising 31% of cases. [3]

Asymptotics

If $2n$ instead of 100 prisoners are considered, where n an arbitrary natural number, the prisoners' survival probability with the cycle-following strategy is given by

$$1 - (H_{2n} - H_n) = 1 - (H_{2n} - \ln 2n) + (H_n - \ln n) - \ln 2$$

With the Euler Mascheroni constant γ for $n \to \infty$

$$\lim_{n\to\infty} (H_n - \ln n) = \gamma$$

holds, which results in an asymptotic survival probability of

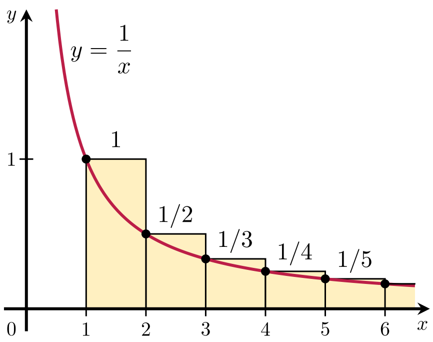


Figure 4: The harmonic numbers are approximately given by the area under the hyperbola and can therefore be approximated by a logarithm. Source: [8]

$$\lim_{n\to\infty} (1 - H_{2n} + H_n) = 1 - \gamma + \gamma - \ln 2 = 1 - \ln 2 \approx 0.30685$$

Since the sequence of probabilites is monotonically decreasing, the prisoners survive with the cycle-following strategy in more than 30% of cases independently of the number of prisoners. [3]

Optimality

In 2006, Eugene Curtin and Max Warshauer gave a proof for the optimality of the cycle-following strategy. The proof is based on an equivalence to a related problem in which all prisoners are allowed to be present in the room and observe the opening of the drawers. This equivalence is based on the correspondence of the (normalized) cycle notation and the one-line notation of permutations. In the second problem, the survival probability is independent of the chosen strategy and equal to the survival probability in the original problem with the cycle-following strategy. Since an arbitrary strategy for the original problem can also be applied to the second problem, but cannot attain a higher survival probability there, the cycle-following strategy has to be optimal. [2]

Bonus problems

B1. The King's Wise Men [13]

Problem

The King called the three wisest men in the country to his court to decide who would become his new advisor. He placed a hat on each of their heads, such that each wise man could see all of the other hats, but none of them could see their own. Each hat was either white or blue. The king gave his word to the wise men that at least one of them was wearing a blue hat - in other words, there could be one, two, or three blue hats, but not zero. The king also announced that the contest would be fair to all three men. The wise men were also forbidden to speak to each other. The king declared that whichever man stood up first and announced the color of his own hat would become his new advisor. The wise men sat for a very long time before one stood up and correctly announced the answer. What did he say, and how did he work it out?

Solution

This is one of the simplest *induction puzzles* and one of the clearest indicators to the method used. Suppose that there were one blue hat. The person with that hat would see two white hats, and since the king specified that there is at least one blue hat, that wise man would immediately know the color of his hat. However, the other two would see one blue and one white hat and would not be able to immediately infer any information from their observations. Therefore, this scenario would violate the king's specification that the contest would be fair to each. So there must be at least two blue hats.

Suppose then that there were two blue hats. Each wise man with a blue hat would see one blue and one white hat. Since they have already realized that there must be at least two blue hats, they would then immediately know that each were wearing a blue hat. However, the man with the white hat would see two blue hats and would not be able to immediately infer any information from his observations. This scenario, then, would also violate the specification that the contest would be fair to each. So there must be three blue hats.

Since there must be three blue hats, the first man to figure that out will stand up and say blue.

B2. Two statisticians in the woods [1]

Hypothesis

If two statisticians were to lose each other in an infinite forest, the first thing they would do is get drunk. That way, they would walk more or less randomly, which would give them the best chance of finding each other. However, the statisticians should stay sober if they want to pick mushrooms. Stumbling around drunk and without purpose would reduce the area of exploration, and make it more likely that the seekers would return to the same spot, where the mushrooms are already gone.

Solution

An elaborate text on this problem is out of scope of this document. You can find it in the bibliography[1].

Presentation: `http://oskopek.com/2015_popmath_gymy/eng_presentation.pdf`

Exercises with solutions: `http://oskopek.com/2015_popmath_gymy/eng_exercises.pdf`

References

[1] Rhett Allain. *How Should Two Lost People Find Each Other?* 2013. URL: http://www.wired.com/2013/09/two-statisticians-lost-in-the-woods/.

[2] Max Warshauer Eugene Curtin. "The locker puzzle". In: *Mathematical Intelligencer* (2006).

[3] Richard P. Stanley. *Algebraic Combinatorics: Walks, Trees, Tableaux, and More.* 2014.

[4] Wikipedia. *100 prisoners problem.* 2015. URL: http://en.wikipedia.org/wiki/100_prisoners_problem.

[5] Wikipedia. *7 bridges.* 2015. URL: http://commons.wikimedia.org/wiki/File:7_bridges.svg.

[6] Wikipedia. *Graph representation of the permutation (1 3 7 4 5 8 2)(6).* 2015. URL: http://en.wikipedia.org/wiki/File:Permutation_cycles_qtl2.svg.

[7] Wikipedia. *Graph representation of the permutation (1 7 5)(2 4 8)(3 6).* 2015. URL: http://en.wikipedia.org/wiki/File:Permutation_cycles_qtl1.svg.

[8] Wikipedia. *Illustration of the integral test in calculus.* 2015. URL: http://en.wikipedia.org/wiki/File:Integral_Test.svg.

[9] Wikipedia. *Konigsberg bridges.* 2015. URL: http://en.wikipedia.org/wiki/File:Konigsberg_bridges.png.

[10] Wikipedia. *Konigsberg graph.* 2015. URL: http://en.wikipedia.org/wiki/File:K%C3%B6nigsberg_graph.svg.

[11] Wikipedia. *Probability mass function of the length of the longest cycle of a random permutation of length 100. Cycle lengths <= 50 are green, cycle lengths > 50 are red.* 2015. URL: http://en.wikipedia.org/wiki/File:Permutation_longest_cycle_length_pmf_qtl2.svg.

[12] Wikipedia. *Seven Bridges of Königsberg.* 2015. URL: http://en.wikipedia.org/wiki/Seven_Bridges_of_K%C3%B6nigsberg.

[13] Wikipedia. *The King's Wise Men.* 2015. URL: http://en.wikipedia.org/wiki/Induction_puzzles#Examples.